Chapter 4: Equations

Equation 4.1:

$$\hat{y}_i = a + bx_i$$

Equation 4.2:

$$e_i = y_i - \hat{y}_i$$

Equation 4.3:

$$y_i = \hat{y} + e_i$$

Equation 4.4:

$$y_i = a + bx_i + e_i$$

Equation 4.5:

$$e_i = y_i - a - bx_i$$

Equation 4.6:

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - \hat{y}_i \right)^2$$

Equation 4.7:

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - a - bx_i \right)^2$$

Equation 4.8:

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial a}$$

Equation 4.9:

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial a} = \sum_{i=1}^{n} \frac{\partial e_i^2}{\partial a}$$

Equation 4.10:

$$\sum_{i=1}^{n} \frac{\partial e_i^2}{\partial a} = \sum_{i=1}^{n} \frac{\partial\left(y_i - a - bx_i\right)^2}{\partial a}$$

Equation 4.11:

$$\frac{\partial\left(y_i - a - bx_i\right)^2}{\partial a} = -2\left(y_i - a - bx_i\right)$$

Equation 4.12:

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial a} = \sum_{i=1}^{n}\left(-2\left(y_i - a - bx_i\right)\right) = -2\sum_{i=1}^{n}\left(y_i - a - bx_i\right)$$

Equation 4.13:

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial b} = \sum_{i=1}^{n} \frac{\partial e_i^2}{\partial b} = \sum_{i=1}^{n} \frac{\partial\left(y_i - a - bx_i\right)^2}{\partial b}$$

Equation 4.14:

$$\frac{\partial\left(y_i - a - bx_i\right)^2}{\partial b} = -2\left(y_i - a - bx_i\right)x_i$$

Equation 4.15:

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial b} = \sum_{i=1}^{n}\left(-2\left(y_i - a - bx_i\right)x_i\right) = -2\sum_{i=1}^{n}\left(y_i - a - bx_i\right)x_i$$

Equation 4.16:

$$0 = -2\sum_{i=1}^{n}\left(y_i - a - bx_i\right)$$

Equation 4.17:

$$0 = \sum_{i=1}^{n}\left(y_i - a - bx_i\right)$$

Equation 4.18:

$$0 = \sum_{i=1}^{n}\left(y_i - a - bx_i\right)x_i$$

Equation 4.19:

$$0 = \sum_{i=1}^{n} \left(y_i - a - bx_i \right) = \sum_{i=1}^{n} e_i$$

Equation 4.20:

$$\overline{e} = \frac{\sum_{i=1}^{n} e_i}{n} = \frac{0}{n} = 0$$

Equation 4.21:

$$0 = \sum_{i=1}^{n} \left(y_i - a - bx_i \right) x_i = \sum_{i=1}^{n} e_i x_i$$

Equation 4.22:

$$0 = \overline{x} \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} e_i \overline{x}$$

Equation 4.23:

$$0 = \sum_{i=1}^{n} e_i x_i - \sum_{i=1}^{n} e_i \overline{x}$$

Equation 4.24:

$$0 = \sum_{i=1}^{n} \left(e_i x_i - e_i \overline{x} \right) = \sum_{i=1}^{n} e_i \left(x_i - \overline{x} \right)$$

Equation 4.25:

$$e_i = e_i - \overline{e}$$

Equation 4.26:

$$0 = \sum_{i=1}^{n} e_i \left(x_i - \overline{x} \right) = \sum_{i=1}^{n} \left(e_i - \overline{e} \right) \left(x_i - \overline{x} \right)$$

Equation 4.27:

$$0 = \frac{\sum_{i=1}^{n} \left(e_i - \overline{e} \right) \left(x_i - \overline{x} \right)}{n-1}$$

Equation 4.28:

$$0 = \mathrm{COV}\left(e, X \right) = \mathrm{CORR}\left(e, X \right)$$

Equation 4.29:

$$0 = \sum_{i=1}^{n} \left(y_i - a - bx_i \right) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - \sum_{i=1}^{n} bx_i$$

Equation 4.30:

$$\sum_{i=1}^{n} a = na$$

Equation 4.31:

$$\sum_{i=1}^{n} bx_i = b\sum_{i=1}^{n} x_i$$

Equation 4.32:

$$0 = \sum_{i=1}^{n} y_i - na - b\sum_{i=1}^{n} x_i$$

Equation 4.33:

$$0 = \overline{y} - a - b\overline{x}$$

Equation 4.34:

$$\overline{y} = a + b\overline{x}$$

Equation 4.35:

$$a = \overline{y} - b\overline{x}$$

Equation 4.36:

$$0 = \sum_{i=1}^{n} \left(\left(y_i - \overline{y} \right) - \left(bx_i - b\overline{x} \right) \right) x_i = \sum_{i=1}^{n} \left(\left(y_i - \overline{y} \right) - b\left(x_i - \overline{x} \right) \right) x_i$$

Equation 4.37:

$$b = \frac{\sum_{i=1}^{n} (y_i - \overline{y}) x_i}{\sum_{i=1}^{n} (x_i - \overline{x}) x_i}$$

Equation 4.38:

$$\hat{y}_i + \Delta y = a + b(x_i + \Delta x)$$

Equation 4.39:

$$\hat{y}_i + \Delta y = (a + bx_i) + b\Delta x = \hat{y}_i + b\Delta x$$

Equation 4.40:

$$b = \frac{\sum_{i=1}^{n} (y_i - \overline{y}) x_i}{\sum_{i=1}^{n} (x_i - \overline{x}) x_i} = \frac{\sum_{i=1}^{n} (y_i - \overline{y})(x_i - \overline{x})}{\sum_{i=1}^{n} (x_i - \overline{x})^2}$$

Equation 4.41:

$$b = \frac{\sum_{i=1}^{n} (y_i - \overline{y})(x_i - \overline{x})}{\sum_{i=1}^{n} (x_i - \overline{x})^2} = \frac{\left(\frac{\sum_{i=1}^{n} (y_i - \overline{y})(x_i - \overline{x})}{n-1} \right)}{\left(\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n-1} \right)}$$

Equation 4.42:

$$b = \frac{\mathrm{COV}(X,Y)}{\mathrm{V}(X)}$$

Equation 4.43:

$$b = \frac{\mathrm{COV}(X,Y)}{\mathrm{SD}(X)\mathrm{SD}(Y)}\frac{\mathrm{SD}(Y)}{\mathrm{SD}(X)}$$

Equation 4.44:

$$b = \mathrm{CORR}(X,Y)\frac{\mathrm{SD}(Y)}{\mathrm{SD}(X)}$$

Equation 4.45:

$$y_i - \overline{y}$$

Equation 4.46:

$$y_i - \overline{y} = (y_i - \hat{y}_i) + (\hat{y}_i - \overline{y})$$

Equation 4.47:

$$\begin{aligned}(y_i - \overline{y})^2 &= \left((y_i - \hat{y}_i) + (\hat{y}_i - \overline{y})\right)^2 \\ &= (y_i - \hat{y}_i)^2 + (\hat{y}_i - \overline{y})^2 + 2(y_i - \hat{y}_i)(\hat{y}_i - \overline{y})\end{aligned}$$

Equation 4.48:

$$\sum_{i=1}^{n}\left(y_i-\overline{y}\right)^2=\sum_{i=1}^{n}\left(y_i-\hat{y}_i\right)^2+\sum_{i=1}^{n}\left(\hat{y}_i-\overline{y}\right)^2+2\sum_{i=1}^{n}\left(y_i-\hat{y}_i\right)\left(\hat{y}_i-\overline{y}\right)$$

Equation 4.49:

$$\hat{y}_i-\overline{y}=\left(a+bx_i\right)-\left(a+b\overline{x}\right)=bx_i-b\overline{x}=b\left(x_i-\overline{x}\right)$$

Equation 4.50:

$$2\sum_{i=1}^{n}\left(y_i-\hat{y}_i\right)\left(\hat{y}_i-\overline{y}\right)=2\sum_{i=1}^{n}e_ib\left(x_i-\overline{x}\right)$$

Equation 4.51:

$$2\sum_{i=1}^{n}e_ib\left(x_i-\overline{x}\right)=2b\left(\sum_{i=1}^{n}e_ix_i-\sum_{i=1}^{n}e_i\overline{x}\right)$$

Equation 4.52:

$$2b\left(\sum_{i=1}^{n}e_ix_i-\sum_{i=1}^{n}e_i\overline{x}\right)=2b\left(\sum_{i=1}^{n}e_ix_i-\overline{x}\sum_{i=1}^{n}e_i\right)$$

Equation 4.53:

$$\frac{\sum_{i=1}^{n}\left(y_i-\overline{y}\right)^2}{n-1}=\frac{\sum_{i=1}^{n}e_i^2}{n-1}+\frac{\sum_{i=1}^{n}\left(\hat{y}_i-\overline{y}\right)^2}{n-1}$$

Equation 4.54:

$$\frac{\sum_{i=1}^{n}\left(\hat{y}_i-\bar{y}\right)^2}{n-1}=\frac{\sum_{i=1}^{n}\left(b\left(x_i-\bar{x}\right)\right)^2}{n-1}=\frac{b^2\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{n-1}=b^2\mathrm{V}(X)$$

Equation 4.55:

$$1=\frac{\left(\frac{\sum_{i=1}^{n}e_i^2}{n-1}\right)}{\left(\frac{\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2}{n-1}\right)}+\frac{\left(\frac{b^2\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{n-1}\right)}{\left(\frac{\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2}{n-1}\right)}$$

Equation 4.56:

$$1=\frac{\sum_{i=1}^{n}e_i^2}{\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2}+\frac{b^2\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2}$$

Equation 4.57:

$$R^2=1-\frac{\sum_{i=1}^{n}e_i^2}{\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2}=\frac{b^2\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2}$$

Equation 4.58:

$$R^2=\left(\mathrm{CORR}\left(X,Y\right)\right)^2$$

Equation 4.59:

$$R^2 = \left(\text{CORR}\left(Y, \hat{Y}\right)\right)^2$$

Equation 4.60:

$$\text{adjusted } R^2 = 1 - \frac{\left(\frac{\sum_{i=1}^{n} e_i^2}{n-2}\right)}{\left(\frac{\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2}{n-1}\right)} = 1 - \frac{\sum_{i=1}^{n} e_i^2}{\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2}\left(\frac{n-1}{n-2}\right)$$

Equation 4.61:

$$R^2 > \text{adjusted } R^2$$

Equation 4.62:

$$\frac{\partial\left(y_i - a - bx_i\right)^2}{\partial a} = \frac{\partial\left(y_i - a - bx_i\right)^2}{\partial\left(y_i - a - bx_i\right)} \frac{\partial\left(y_i - a - bx_i\right)}{\partial a}$$

Equation 4.63:

$$\frac{\partial\left(y_i - a - bx_i\right)^2}{\partial\left(y_i - a - bx_i\right)} = 2\left(y_i - a - bx_i\right)^{2-1} = 2\left(y_i - a - bx_i\right)$$

Equation 4.64:

$$\frac{\partial\left(y_i - a - bx_i\right)}{\partial a} = \frac{\partial\left(-a\right)}{\partial a}$$

Equation 4.65:

$$\frac{\partial\left(-a\right)}{\partial a} = -\frac{\partial a}{\partial a}$$

Equation 4.66:

$$-\frac{\partial a}{\partial a} = -1$$

Equation 4.67:

$$\frac{\partial\left(y_i - a - bx_i\right)^2}{\partial b} = \frac{\partial\left(y_i - a - bx_i\right)^2}{\partial\left(y_i - a - bx_i\right)} \frac{\partial\left(y_i - a - bx_i\right)}{\partial b}$$

Equation 4.68:

$$\frac{\partial\left(y_i - a - bx_i\right)}{\partial b} = \frac{\partial\left(-bx_i\right)}{\partial b}$$

Equation 4.69:

$$\frac{\partial\left(-bx_i\right)}{\partial b} = -x_i$$

Equation 4.70:

$$\frac{\partial}{\partial a}\left(\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial a}\right)=\frac{\partial^2\left(\sum_{i=1}^{n} e_i^2\right)}{\partial a^2}=\frac{\partial}{\partial a}\left(-2\sum_{i=1}^{n}\left(y_i-a-bx_i\right)\right)$$

Equation 4.71:

$$\frac{\partial^2\left(\sum_{i=1}^{n} e_i^2\right)}{\partial a^2}=-2\frac{\partial}{\partial a}\left(\sum_{i=1}^{n}\left(y_i-a-bx_i\right)\right)=-2\sum_{i=1}^{n}\frac{\partial\left(y_i-a-bx_i\right)}{\partial a}$$

Equation 4.72:

$$\frac{\partial^2\left(\sum_{i=1}^{n} e_i^2\right)}{\partial a^2}=-2\sum_{i=1}^{n}\frac{\partial\left(y_i-a-bx_i\right)}{\partial a}=-2\sum_{i=1}^{n}\left(-1\right)=2n$$

Equation 4.73:

$$\frac{\partial}{\partial b}\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial b}=\frac{\partial\left[-2\sum_{i=1}^{n}\left(y_i-a-bx_i\right)x_i\right]}{\partial b}.=-2\sum_{i=1}^{n}\frac{\partial\left[\left(y_i-a-bx_i\right)x_i\right]}{\partial b}$$

Equation 4.74:

$$\frac{\partial\left(-bx_i^2\right)}{\partial x_i}=-x_i^2$$

Equation 4.75:

$$\frac{\partial}{\partial b}\frac{\partial\left(\sum_{i=1}^{n}e_i^2\right)}{\partial b}=\frac{\partial^2\left(\sum_{i=1}^{n}e_i^2\right)}{\partial b^2}=-2\sum_{i=1}^{n}\left(-x_i^2\right)=2\sum_{i=1}^{n}x_i^2$$

Equation 4.76:

$$\frac{\partial}{\partial b}\left(\frac{\partial\left(\sum_{i=1}^{n}e_i^2\right)}{\partial a}\right)=-2\sum_{i=1}^{n}\left(-x_i\right)=2\sum_{i=1}^{n}x_i$$